数理解析研究所講究録
第 1912 巻 2014 年 95-102

Valuation of Callable and Putable Bonds under the Generalized Ho–Lee model: A Stochastic Game Approach

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1 Introduction

Callable bonds are bonds which have the possibility that the issuer may do prepayment before its maturity for own convenience. The issuer is usually allowed to early exercise a call option in coupon times before its maturity. Therefore, it is an American–type interest rate derivative. If the issuer exercises the right, she purchases back the bond from an investor for a call price. Then, the investor suffers a loss because of prepayment. We generally call the possible loss "prepayment risk." While, putable bonds are bonds with which an investor can demand prepayment for the issuer before its maturity. If an investor exercises her right, the issuer must purchase the bond for a put price. In this paper, we consider the bond where the issuer and investors simultaneously have callable and putable options, respectively. To take prepayment risk into consideration, we assume that the call and put prices are dependent on time and states.

As an interest rate model, we consider the Generalized Ho–Lee model ([2, 3, 4]). This model is a discrete–time and binomial lattice interest rate model. Since the volatilities are dependent on time and states, it is flexible to model realistic interest rate processes. Moreover, it has the term structure of interest rate at all nodes on the lattice. We assume that the issuer and an investor decide whether or not to exercise their option by taking account of the interest rate market.

The theory of stochastic games was originated by the seminal paper of Shapley (1953) [9], and it can handle finite and infinite time horizon problems. We apply the stochastic game approach to the valuation of the callable and putable bond as a finite time horizon problem. That is, in each exercisable time and a state, we consider that the investor and issuer face a zero–sum game whose payoff structure is dependent on the interest rate market. Given these assumptions, using a dynamic programming approach, we can derive the optimality equation for valuation of the bond.

The organization of this paper is as follows. In Section 2, we illustrate the generalized Ho–Lee model. In Section 3, we define the callable and putable bonds. Then we derive the optimality equation to evaluate the bond values. In Section 4, we show numerical examples, where we consider the influence on the issuer's optimal exercise strategies for the change in the call price. Section 5 contains a conclusion.

2 The Generalized Ho–Lee Model

The Generalized Ho–Lee model is a binomial lattice interest rate model ([2, 3, 4]). A node on the lattice is represented by (n,i) where n denotes the time and i the state for $0 \le n \le N$ and $0 \le i \le n$, respectively, N is the time horizon of the model. Let $P(n,i;T)$ represent the zero–coupon bond price at node (n,i) with remaining maturity of T periods.

We call $\delta(n,i;T)$ the binomial volatilities, which stand for uncertainty of interest rate on the binomial lattice. We define it as

$$\delta(n,i;1) = \frac{P(n+1,i+1;1)}{P(n+1,i;1)}.$$

The one–period binomial volatilities in the Generalized Ho–Lee model are given by

$$\delta(n,i;1) = \exp\Big(-2\sigma(n)\min\big(R(n,i;1),R\big)\Delta t^{3/2}\Big), \tag{2.1}$$

where $\sigma(n)$ denotes an interest rate volatilities, $R(n,i;1)$ the one–period yield, R a thereshold rate, and Δt a time interval of one period. Moreover, the T–periods binomial volatilities in the model are given by

$$\delta(n,i;T) = \delta(n,i;1)\delta(n+1,i;T-1)\left(\frac{1+\delta(n+1,i+1;T-1)}{1+\delta(n+1,i;T-1)}\right). \tag{2.2}$$

Equation (2) shows the arbitrage free condition in the Generalized Ho–Lee model. To satisfy Equation (2), the one–period zero–coupon bond prices in this model are given by

$$P(n,i;1) = \frac{P(0,0;n+1)}{P(0,0;n)}\prod_{k=1}^{n}\left(\frac{1+\delta(k-1,0;n-k)}{1+\delta(k-1,0;n-k+1)}\right)\prod_{j=0}^{i-1}\delta(n-1,j;1), \tag{2.3}$$

and the zero–coupon bond prices with the remaining maturity of T periods are similarly given by

$$P(n,i;T) = \frac{P(0,0;n+T)}{P(0,0;n)}\prod_{k=1}^{n}\left(\frac{1+\delta(k-1,0;n-k)}{1+\delta(k-1,0;n-k+T)}\right)\prod_{j=0}^{i-1}\delta(n-1,j;T). \tag{2.4}$$

Solving recursively Equations (2.1)–(2.4), we can derive the term structure of the interest rate for any node (n,i).

3 A Stochastic Game Approach

3.1 Callable and Putable Bonds

Let $t_0, t_1, \ldots, t_N$ be the time sequences, where t_0 is the initial time, $t_1, t_2, \ldots, t_{N-1}$ are the coupon times and t_N is the maturity time. The coupons of the bond are represented by c.

Callable bonds are bonds that give the issuer the right to purchase back the bond for a call price at the admissible exercise times within the bond's life. Similarly, putable bonds are bonds that give investors the right to demand for the issuer to purchase the bond for a put price at the admissible exercise times within the bond's life.

Each of the investor and issuer decides whether or not to exercise the right at the coupon times t_n $(n \in \{n^*, n^*+1, \ldots, N-1\})$, where t_{n^*} is the first admissible exercise time. If each player (or both players) exercises the right at the exercisable time t_n, the investor receives the payoffs from the issuer at the next coupon time t_{n+1}. For simplicity, we use a time index n to represent the time, instead of t_n.

Let $C(n,i)$ and $P(n,i)$ denote the call and put prices at node (n,i), respectively. We usually have $P(n,i) \leq C(n,i)$. If both players simultaneously exercise the right then the investor receives a payoff $\varphi(n,i)$ from the issuer. We assume that $P(n,i) \leq \varphi(n,i) \leq C(n,i)$.

3.2 A Stochastic Game Approach

We consider the situation where the issuer and an investor are simultaneously granted the right to exercise the call and put options, respectively. The investor and issuer choose the strategy x and y ($x, y \in \{Exercise, Not\,Exercise\} =: S$) at each exercisable node (n,i), respectively. The investor gains the payoff $\mathcal{A}(x,y;n,i)$ from the issuer whenever the pair x, y is chosen at node (n,i). The payoffs of a stage game at node (n,i) are given by

$$\mathcal{A}(x,y;n,i) = \begin{cases} C(n,i) + c & \text{if the issuer exercises,} \\ P(n,i) + c & \text{if the investor exercises,} \\ \varphi(n,i) + c & \text{if both players exercise.} \end{cases}$$

If both players hold the bond, the game is carried over to the next time t_{n+1}. Under this situation, the two players face a two–person, zero–sum game whose payoff structure is dependent on node (n,i) composed of interest rate market. We call it a stochastic game, or a Markov game.

For a two–person, zero–sum game (a matrix game) defined by a payoff matrix $A \in \mathbb{R}^{m\times n}$ $(m, n \in \mathbb{N} := \{1, 2, \cdots\})$, we define the value of the game as follows:

$$val[A] := \min_{\boldsymbol{q}\in\Delta^n} \max_{\boldsymbol{p}\in\Delta^m} \boldsymbol{p}^\top A\boldsymbol{q} = \max_{\boldsymbol{p}\in\Delta^m} \min_{\boldsymbol{q}\in\Delta^n} \boldsymbol{p}^\top A\boldsymbol{q}, \tag{3.1}$$

where, for a positive integer $\ell \in \mathbb{N}$, Δ^ℓ denotes the ℓ–dimensional unit simplex defined by:

$$\Delta^\ell := \left\{ \boldsymbol{x} = (x_1, \ldots, x_\ell)^\top \in \mathbb{R}^\ell : \ x_i \geq 0, \ i = 1, \ldots, \ell; \ \sum_{i=1}^{\ell} x_i = 1 \right\}.$$

Let $\rho(n,i;1)$ denote the one–period discount factor at node (n,i), which is the one–period zero–coupon bond price calculated by the Generalized Ho–Lee model, and $V(n,i)$ the bond value at node (n,i). Then, according to the optimality principle, by solving the following optimality equation backwardly in n, we can simultaneously derive the bond value at the initial time and the optimal exercise strategies of the investor and issuer.

Optimality Equation:

$$V(n,i) = F + c = 1 + c \tag{3.2}$$
$$\text{for } n = N, i \in \{0, 1, \ldots, N\}$$

$$V(n,i) = \rho(n,i;1)\left\{ val \begin{bmatrix} \varphi(n,i) & P(n,i) \\ C(n,i) & \mathbb{E}^{\mathbb{Q}}[V(n+1, I_{n+1})] \end{bmatrix} + c \right\} \tag{3.3}$$
$$\text{for } n = N-1, N-2, \ldots, n^*$$

$$V(n,i) = \rho(n,i;1)\left\{ \mathbb{E}^{\mathbb{Q}}[V(n+1, I_{n+1})] + c \right\} \tag{3.4}$$
$$\text{for } n = n^* - 1, n^* - 2, \ldots, 0,$$

where F denotes the principal of the bond which is scaled to 1, I_{n+1} a random state in the next coupon time t_{n+1}, and $\mathbb{Q}$ the risk–neutral probability measure.

Next, we prove the equilibrium solution at the exercisable node (n,i).

Theorem 1. *For the all exercisable node (n,i), the matrix game possess saddle points in the pure strategies:*

$$\max_{x\in S} \min_{y \in S} \mathcal{A}(x,y;n,i) = \min_{y \in S} \max_{x \in S} \mathcal{A}(x,y;n,i).$$

Furthermore, the optimal exercise strategies are given by

$$(x,y) = \begin{cases} (E,N) & \text{if } \mathbb{E}^{\mathbb{Q}}[V(n+1,I_{n+1})] \le P(n,i) \le C(n,i) \\ (N,N) & \text{if } P(n,i) \le \mathbb{E}^{\mathbb{Q}}[V(n+1,I_{n+1})] \le C(n,i) \\ (N,E) & \text{if } P(n,i) \le C(n,i) \le \mathbb{E}^{\mathbb{Q}}[V(n+1,I_{n+1})], \end{cases}$$

where E and N denote the strategy "Exercise" and "Not Exercise", respectively.

Proof. The issuer chooses the strategy N when $\mathbb{E}^{\mathbb{Q}}[V(n+1,I_{n+1})] \le P(n,i) \le C(n,i)$, because N is the weakly dominant strategy for the issuer (the minimizer). The investor then chooses the best response strategy E and hence $(x,y) = (E,N)$. When $P(n,i) \le \mathbb{E}^{\mathbb{Q}}[V(n+1,I_{n+1})] \le C(n,i)$, for both players, the strategy N is the weakly dominant strategy and $(x,y) = (N,N)$. When $P(n,i) \le C(n,i) \le \mathbb{E}^{\mathbb{Q}}[V(n+1,I_{n+1})]$, the investor chooses the strategy N which is the weakly dominant strategy for him (the maximizer). Then, the issuer chooses the best response strategy E, and hence $(x,y) = (N,E)$. □

4 Numerical Examples

In this section, we show numerical examples on the basis of the above argument. For the interest rate calculated by the Generalized Ho–Lee model, we propose the values for the simple fixed coupon bond and the bond including the callable and putable options. Thereby, we discuss the bond values in the initial time and the optimal exercise strategies of the issuer.

We set the time sequences $t_0, t_1, \ldots, t_{20}$, the exercisable times $t_{10}, t_{11}, \ldots, t_{19}$, and $\Delta(t) = 0.5$. That is, we consider the bond that the maturity is 10–year and the admissible exercise times are from 5–year onward. The other parameters are a coupon $c = 0.1$ and the interest rate volatilities in the Generalized Ho–Lee model $\sigma(n) = 0.2$ which is constant in n. We set the call price $C(n, i) = 1.4$, 1.6, 1.8 and the put price $P(n, i) = 1$.

Table 1 shows the values of the 10–year fixed coupon bond, and Tables 2, 3 and 4 shows the values of the bond including the options for the call price 1.4, 1.6 and 1.8, respectively. In all Tables, the upside of Table means high interest rates. For Tables 2–4, the upper surrounded area represents the investor's exercise nodes and the lower the issuer's exercise nodes. The parameters in Tables 2–4 are the same values except for a call price.

Tables 2–4 show that the investor exercises the right in high interest rates while the issuer exercises the right in low interest rates. Accordingly, if interest rates is high, we can infer that the investor may do prepayment to seek the higher yields on an investment. While, if interest rates is low, we can similarly infer that the issuer may do prepayment to switch to loans with lower interest rates.

As for the bond values in the initial time, the fixed coupon bond in Table 1 is the highest of the four ones. Furthermore, with regard to bonds including the options, Tables 2–4 show that as the call price is higher, the bond values in the initial time are higher.

In order to consider the investor's prepayment risk for a change in the call price, we observe the issuer's optimal exercise strategies. Tables 2–4 show that as the call prices are lower, the issuer's optimal exercise area is bigger. This is because, if we regard a call price as a penalty for the issuer doing prepayment before the maturity, the issuer has the more incentive to exercise the right as the call price is lower.

																				1.1
																			0.897	1.1
																		0.782	0.971	1.1
																	0.721	0.897	1.028	1.1
																0.693	0.860	0.991	1.073	1.1
															0.689	0.848	0.979	1.067	1.106	1.1
														0.701	0.856	0.986	1.078	1.126	1.131	1.1
													0.726	0.878	1.008	1.103	1.157	1.171	1.150	1.1
												0.762	0.913	1.042	1.139	1.197	1.218	1.205	1.164	1.1
											0.807	0.957	1.086	1.184	1.246	1.272	1.265	1.231	1.174	1.1
										0.860	1.010	1.139	1.238	1.303	1.333	1.330	1.301	1.249	1.181	1.1
									0.921	1.071	1.200	1.300	1.366	1.398	1.400	1.375	1.327	1.263	1.186	1.1
								0.990	1.140	1.269	1.368	1.435	1.469	1.473	1.451	1.408	1.347	1.274	1.190	1.1
							1.065	1.215	1.344	1.443	1.510	1.545	1.551	1.531	1.490	1.432	1.362	1.281	1.193	1.1
						1.147	1.298	1.425	1.523	1.589	1.625	1.632	1.614	1.575	1.519	1.451	1.372	1.286	1.195	1.1
					1.235	1.386	1.512	1.609	1.674	1.709	1.716	1.699	1.662	1.608	1.541	1.464	1.380	1.290	1.197	1.1
				1.329	1.480	1.604	1.699	1.763	1.797	1.804	1.787	1.751	1.698	1.632	1.557	1.474	1.386	1.293	1.198	1.1
			1.430	1.579	1.702	1.794	1.856	1.889	1.895	1.878	1.841	1.789	1.725	1.651	1.569	1.481	1.390	1.295	1.198	1.1
		1.536	1.684	1.804	1.893	1.953	1.983	1.988	1.970	1.934	1.883	1.819	1.745	1.664	1.577	1.487	1.393	1.297	1.199	1.1
	1.647	1.793	1.910	1.996	2.053	2.081	2.084	2.065	2.029	1.977	1.913	1.840	1.760	1.674	1.584	1.490	1.395	1.298	1.199	1.1
1.763	1.906	2.020	2.103	2.156	2.181	2.182	2.162	2.125	2.073	2.009	1.936	1.856	1.771	1.681	1.588	1.493	1.396	1.298	1.199	1.1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

Time

Table 4.1: Values of the 10–year fixed coupon bond

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
																				1.1
																			0.897	1.1
																		0.832	0.971	1.1
																	0.844	0.897	1.028	1.1
																0.857	0.906	0.991	1.073	1.1
															0.870	0.915	0.979	1.067	1.106	1.1
														0.884	0.926	0.986	1.078	1.126	1.131	1.1
													0.898	0.936	1.008	1.103	1.157	1.171	1.150	1.1
												0.912	0.947	1.042	1.139	1.197	1.218	1.205	1.164	1.1
											0.925	0.972	1.086	1.184	1.246	1.272	1.265	1.231	1.174	1.1
										0.938	1.017	1.139	1.238	1.303	1.333	1.330	1.301	1.249	1.181	1.1
									0.956	1.074	1.200	1.300	1.366	1.398	1.400	1.375	1.327	1.263	1.186	1.1
								1.005	1.140	1.265	1.360	1.418	1.435	1.449	1.451	1.408	1.347	1.274	1.190	1.1
							1.067	1.206	1.322	1.399	1.424	1.439	1.452	1.462	1.471	1.432	1.362	1.281	1.193	1.1
						1.134	1.268	1.370	1.427	1.430	1.444	1.455	1.464	1.472	1.478	1.451	1.372	1.286	1.195	1.1
					1.204	1.330	1.421	1.468	1.473	1.448	1.458	1.467	1.474	1.479	1.484	1.464	1.380	1.290	1.197	1.1
				1.277	1.395	1.477	1.520	1.526	1.505	1.461	1.469	1.475	1.481	1.485	1.488	1.474	1.386	1.293	1.198	1.1
			1.353	1.464	1.540	1.580	1.588	1.569	1.529	1.471	1.477	1.482	1.486	1.489	1.492	1.481	1.390	1.295	1.198	1.1
		1.433	1.539	1.611	1.649	1.657	1.640	1.602	1.547	1.479	1.483	1.487	1.490	1.492	1.494	1.487	1.393	1.297	1.199	1.1
	1.518	1.620	1.688	1.725	1.733	1.717	1.680	1.627	1.561	1.484	1.488	1.490	1.492	1.494	1.496	1.490	1.395	1.298	1.199	1.1
1.608	1.706	1.772	1.806	1.814	1.798	1.762	1.711	1.646	1.571	1.488	1.491	1.493	1.494	1.496	1.497	1.493	1.396	1.298	1.199	1.1

Time

Table 4.2: Values of the bond including the options for the call price 1.4

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
																				1.1
																			0.897	1.1
																		0.832	0.971	1.1
																	0.844	0.897	1.028	1.1
																0.857	0.906	0.991	1.073	1.1
															0.870	0.915	0.979	1.067	1.106	1.1
														0.884	0.926	0.986	1.078	1.126	1.131	1.1
													0.898	0.936	1.008	1.103	1.157	1.171	1.150	1.1
												0.912	0.947	1.042	1.139	1.197	1.218	1.205	1.164	1.1
											0.925	0.972	1.086	1.184	1.246	1.272	1.265	1.231	1.174	1.1
										0.938	1.017	1.139	1.238	1.303	1.333	1.330	1.301	1.249	1.181	1.1
									0.956	1.074	1.200	1.300	1.366	1.398	1.400	1.375	1.327	1.263	1.186	1.1
								1.005	1.141	1.269	1.368	1.435	1.469	1.473	1.451	1.408	1.347	1.274	1.190	1.1
							1.072	1.216	1.344	1.443	1.510	1.545	1.551	1.531	1.490	1.432	1.362	1.281	1.193	1.1
						1.150	1.298	1.425	1.523	1.589	1.625	1.632	1.614	1.575	1.519	1.451	1.372	1.286	1.195	1.1
					1.235	1.383	1.505	1.593	1.641	1.641	1.653	1.662	1.662	1.608	1.541	1.464	1.380	1.290	1.197	1.1
				1.324	1.468	1.583	1.661	1.697	1.693	1.656	1.665	1.672	1.678	1.632	1.557	1.474	1.386	1.293	1.198	1.1
			1.416	1.555	1.661	1.729	1.759	1.753	1.720	1.668	1.674	1.679	1.684	1.651	1.569	1.481	1.390	1.295	1.198	1.1
		1.510	1.643	1.741	1.802	1.827	1.821	1.790	1.740	1.676	1.681	1.685	1.688	1.664	1.577	1.487	1.393	1.297	1.199	1.1
	1.607	1.733	1.824	1.880	1.902	1.895	1.866	1.818	1.756	1.682	1.686	1.689	1.691	1.674	1.584	1.490	1.395	1.298	1.199	1.1
1.707	1.826	1.912	1.963	1.982	1.975	1.946	1.900	1.839	1.767	1.687	1.690	1.692	1.694	1.681	1.588	1.493	1.396	1.298	1.199	1.1

Time

Table 4.3: Values of the bond including the options for the call price 1.6

																				1.1
																			0.897	1.1
																		0.832	0.971	1.1
																	0.844	0.897	1.028	1.1
																0.857	0.906	0.991	1.073	1.1
															0.870	0.915	0.979	1.067	1.106	1.1
														0.884	0.926	0.986	1.078	1.126	1.131	1.1
													0.898	0.936	1.008	1.103	1.157	1.171	1.150	1.1
												0.912	0.947	1.042	1.139	1.197	1.218	1.205	1.164	1.1
											0.925	0.972	1.086	1.184	1.246	1.272	1.265	1.231	1.174	1.1
										0.938	1.017	1.139	1.238	1.303	1.333	1.330	1.301	1.249	1.181	1.1
									0.956	1.074	1.200	1.300	1.366	1.398	1.400	1.375	1.327	1.263	1.186	1.1
								1.005	1.141	1.269	1.368	1.435	1.469	1.473	1.451	1.408	1.347	1.274	1.190	1.1
							1.072	1.216	1.344	1.443	1.510	1.545	1.551	1.531	1.490	1.432	1.362	1.281	1.193	1.1
						1.150	1.298	1.425	1.523	1.589	1.625	1.632	1.614	1.575	1.519	1.451	1.372	1.286	1.195	1.1
					1.236	1.386	1.512	1.609	1.674	1.709	1.716	1.699	1.662	1.608	1.541	1.464	1.380	1.290	1.197	1.1
				1.330	1.480	1.604	1.699	1.763	1.797	1.804	1.787	1.751	1.698	1.632	1.557	1.474	1.386	1.293	1.198	1.1
			1.430	1.579	1.701	1.793	1.854	1.885	1.888	1.864	1.841	1.789	1.725	1.651	1.569	1.481	1.390	1.295	1.198	1.1
		1.535	1.683	1.802	1.890	1.946	1.971	1.967	1.934	1.873	1.879	1.819	1.745	1.664	1.577	1.487	1.393	1.297	1.199	1.1
	1.645	1.790	1.905	1.987	2.038	2.057	2.046	2.009	1.951	1.880	1.884	1.840	1.760	1.674	1.584	1.490	1.395	1.298	1.199	1.1
1.759	1.900	2.009	2.086	2.130	2.143	2.128	2.089	2.032	1.963	1.885	1.888	1.856	1.771	1.681	1.588	1.493	1.396	1.298	1.199	1.1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

Time

Table 4.4: Values of the bond including the options for the call price 1.8

5 Conclusion

In this paper, we propose a pricing method for the bond including the callable and putable options. If we consider the bond that both the issuer and the investor are simultaneously granted the right to exercise the option, then we assume that they face a two–person, zero–sum game whose payoff structure is dependent on node (n, i). Thereby, we formulate such bond's valuation as a stochastic game problem. Furthermore, according to a dynamic programming approach, we derive the optimality equation for such bond's valuation. Consequently, we obtain the bond values and the optimal exercise strategies of the issuer and investor. Besides, we show that the stochastic games have a solution in pure strategies and derive the equilibrium solution in stage games.

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